

## NONCOMMUTATIVE TSEN'S THEOREM IN DIMENSION ONE

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ABSTRACT. Let  $k$  be a field. In this paper, we find necessary and sufficient conditions for a noncommutative curve of genus zero over  $k$  to be a noncommutative  $\mathbb{P}^1$ -bundle. This result can be considered a noncommutative, one-dimensional version of Tsen's theorem. By specializing this theorem, we show that every arithmetic noncommutative projective line is a noncommutative curve, and conversely we characterize exactly those noncommutative curves of genus zero which are arithmetic. We then use this characterization, together with results from [9], to address some problems posed in [4].

## 1. INTRODUCTION

Throughout this paper,  $k$  will denote a field. In [4], the concept of a noncommutative curve of genus zero is defined as a small  $k$ -linear abelian category  $\mathbf{H}$  such that

- each object of  $\mathbf{H}$  is noetherian,
- all morphism and extension spaces in  $\mathbf{H}$  are finite dimensional over  $k$ ,
- $\mathbf{H}$  admits an Auslander-Reiten translation, i.e. an autoequivalence  $\tau$  such that Serre duality  $\mathrm{Ext}_{\mathbf{H}}^1(\mathcal{E}, \mathcal{F}) \cong D \mathrm{Hom}_{\mathbf{H}}(\mathcal{F}, \tau \mathcal{E})$  holds, where  $D(-)$  denotes the  $k$ -dual,
- $\mathbf{H}$  has an object of infinite length, and
- $\mathbf{H}$  has a tilting object.

One motivation for the definition is that if  $C$  is a smooth projective curve of genus zero over  $k$ , then the category of coherent sheaves over  $C$  satisfies these properties. Furthermore, Kussin calls the category  $\mathbf{H}$  *homogeneous* if

- for all simple objects  $\mathcal{S}$  in  $\mathbf{H}$ ,  $\mathrm{Ext}_{\mathbf{H}}^1(\mathcal{S}, \mathcal{S}) \neq 0$ .

If  $\mathbf{H}$  is not homogenous (e.g. if  $\mathbf{H}$  is a weighted projective line) then  $\mathbf{H}$  is birationally equivalent to a homogeneous noncommutative curve of genus zero [4, p. 2]. Therefore, from the perspective of noncommutative birational geometry, the homogeneous curves play a crucial role.

If  $\mathbf{H}$  is a homogenous noncommutative curve of genus zero and  $\mathcal{L}$  is a line bundle on  $\mathbf{H}$ , then there exists an indecomposable bundle  $\overline{\mathcal{L}}$  and an irreducible morphism  $\mathcal{L} \rightarrow \overline{\mathcal{L}}$  coming from an AR sequence starting at  $\mathcal{L}$ . Kussin calls the bimodule  $M := {}_{\mathrm{End}(\overline{\mathcal{L}})} \mathrm{Hom}_{\mathbf{H}}(\mathcal{L}, \overline{\mathcal{L}})_{\mathrm{End}(\mathcal{L})}$  the *underlying bimodule* of  $\mathbf{H}$ . It turns out that the only possibilities for the left-right dimensions of  $M$  are  $(1, 4)$  and  $(2, 2)$ .

On the other hand, in [13], M. van den Bergh introduces the notion of a noncommutative  $\mathbb{P}^1$ -bundle over a pair of commutative schemes  $X, Y$ . In particular, if  $K$  and  $L$  are finite extensions of  $k$  and  $N$  is a  $k$ -central  $K - L$ -bimodule of

finite dimension as both a  $K$ -module and an  $L$ -module, then one can form the  $\mathbb{Z}$ -algebra  $\mathbb{S}^{n.c.}(N)$ , the noncommutative symmetric algebra of  $N$  (see Section 2 for details). The *noncommutative  $\mathbb{P}^1$ -bundle generated by  $N$* ,  $\mathbb{P}^{n.c.}(N)$ , is defined to be the quotient of the category of graded right  $\mathbb{S}^{n.c.}(N)$ -modules modulo the full subcategory of direct limits of right bounded modules. It is natural to ask whether a homogeneous noncommutative curve of genus zero is a noncommutative  $\mathbb{P}^1$ -bundle generated by  $M$ , at least under the necessary condition that  $\text{End}(\mathcal{L})$  and  $\text{End}(\overline{\mathcal{L}})$  are commutative. Our main result is that this is the case. Before we state it precisely, we need to introduce some notation. If  $\mathbf{C}$  is a noetherian category, then there exists a unique locally noetherian category  $\tilde{\mathbf{C}}$  whose full subcategory of noetherian objects is  $\mathbf{C}$  [12, Theorem 2.4]. Furthermore, if  $\mathbf{C}$  and  $\mathbf{D}$  are  $k$ -linear categories and there exists a  $k$ -linear equivalence  $\mathbf{C} \rightarrow \mathbf{D}$ , we write  $\mathbf{C} \equiv \mathbf{D}$ .

Our main theorem is the following (Theorem 3.10):

**Theorem 1.1.** *If  $\mathbf{H}$  is a homogeneous noncommutative curve of genus zero over with underlying bimodule  $M$  such that  $\text{End}(\mathcal{L})$  and  $\text{End}(\overline{\mathcal{L}})$  are commutative, then*

$$\tilde{\mathbf{H}} \equiv \mathbb{P}^{n.c.}(M).$$

*Conversely, if  $K$  and  $L$  are finite extensions of  $k$  and  $N$  is a  $k$ -central  $K - L$ -bimodule of left-right dimension  $(2, 2)$  or  $(1, 4)$ , then  $\mathbb{P}^{n.c.}(N)$  is a noncommutative curve of genus zero with underlying bimodule  $N$ .*

If we think of a homogeneous noncommutative curve of genus zero as being a kind of conic bundle over the pair  $\text{Spec}(\text{End}(\mathcal{L}))$ ,  $\text{Spec}(\text{End}(\overline{\mathcal{L}}))$ , then Theorem 1.1 can be interpreted as saying that all noncommutative one-dimensional conic bundles are noncommutative one-dimensional  $\mathbb{P}^1$ -bundles, whence the connection to Tsen's theorem.

The main idea behind the proof of Theorem 1.1 is that one can form a  $\mathbb{Z}$ -algebra coordinate ring for  $\tilde{\mathbf{H}}$  by employing *all* of the indecomposable bundles in  $\mathbf{H}$ , as opposed to constructing the  $\mathbb{Z}$ -graded algebra using only the  $\tau$ -orbit of  $\mathcal{L}$ , as is done in [4]. One then shows that this  $\mathbb{Z}$ -algebra is isomorphic to the noncommutative symmetric algebra  $\mathbb{S}^{n.c.}(M)$ . Fortunately, many of the technical details needed to construct this isomorphism are provided by [2, Proposition 2.1 and 2.2] and [4].

**1.1. Noncommutative curves of genus zero and arithmetic noncommutative projective lines.** We now describe a specialization of Theorem 1.1 and its applications. In [9], the notion of arithmetic noncommutative projective line over  $k$  is studied. These are none other than spaces of the form  $\mathbb{P}^{n.c.}(V)$  where  $V$  is a  $k$ -central,  $K - K$ -bimodule of left-right dimension  $(2, 2)$ . Among other things, the automorphism groups of these spaces are computed in terms of arithmetic data defining  $V$  [9, Lemma 7.2, Lemma 7.3, and Theorem 7.4], and isomorphism invariants are determined [9, Theorem 5.5]. As a consequence of Theorem 1.1 we have the following (Corollary 3.11):

**Corollary 1.2.** *Suppose  $\mathbf{H}$  is a homogeneous noncommutative curve of genus zero such that  $M$  has left-right dimension  $(2, 2)$ , and  $\text{End}(\mathcal{L})$  and  $\text{End}(\overline{\mathcal{L}})$  are isomorphic and commutative. Then  $\tilde{\mathbf{H}} \equiv \mathbb{P}^{n.c.}(M)$ , so that  $\tilde{\mathbf{H}}$  is an arithmetic noncommutative projective line.*

*Conversely, every arithmetic noncommutative projective line  $\mathbb{P}^{n.c.}(V)$  is a homogeneous noncommutative curve of genus zero with underlying bimodule  $V$ .*

Corollary 1.2 has several applications to both arithmetic noncommutative projective lines and homogeneous noncommutative curves of genus zero, which we now describe.

**1.2. Applications to arithmetic noncommutative projective lines.** In [9, Theorem 3.7], homological techniques are employed to give a proof that the noncommutative symmetric algebra of a  $k$ -central  $K - K$ -bimodule of left-right dimension  $(2, 2)$ ,  $V$ , is a domain in the sense that if  $x \in \mathbb{S}^{n.c.}(V)_{ij}$  and  $y \in \mathbb{S}^{n.c.}(V)_{jl}$  then  $xy = 0$  implies that  $x = 0$  or  $y = 0$ . Using Corollary 1.2, we produce a much shorter proof of a generalization (Proposition 4.2).

In [9], a classification of spaces of the form  $\mathbb{P}^{n.c.}(V)$  up to  $k$ -linear equivalence, and a classification of isomorphisms between such spaces, is described. It is natural, then, to determine whether a Bondal-Orlov theorem exists for arithmetic noncommutative projective lines. It is known that such a theorem holds more generally for homogeneous noncommutative curves of genus zero, and therefore, by Corollary 1.2, we deduce (Theorem 4.3) a Bondal-Orlov theorem for arithmetic noncommutative projective lines. Since we could not find any proofs of this fact in the literature, we provide a proof for the readers convenience.

**1.3. Applications to homogeneous noncommutative curves of genus zero.**

We use Corollary 1.2 to address three questions that appear in [4], which we now describe. In [4, 1.1.5], Kussin partitions the collection of underlying bimodules into three classes, or “orbit cases”. Orbit case I consists of those  $M$  which have left-right dimension  $(1, 4)$ . Orbit case II consists of those  $M$  with left-right dimension  $(2, 2)$  such that the group of point preserving  $k$ -linear autoequivalences acts transitively on the set of line bundles. Orbit case III consists of all other  $M$  of left-right dimension  $(2, 2)$ . Kussin poses a number of problems related to this trichotomy [4]. For example he asks the following

**Question 1.3.** ([4, Problem 1.1.11]): Is there a criterion from which one can easily decide whether a given bimodule of left-right dimension  $(2, 2)$  is in orbit case II or III?

We now describe the next question we address. In certain cases in which  $M$  is a non-simple bimodule of left-right dimension  $(2, 2)$ , and the center of  $\text{End}(\overline{\mathcal{L}})$  has cyclic Galois group over  $k$ , Kussin obtains a precise description of the automorphism group of  $\mathbf{H}$ , denoted  $\text{Aut}(\mathbb{X})$  [4, Theorem 5.3.4]. By definition, this is the group of isomorphism classes of  $k$ -linear equivalences fixing  $\mathcal{L}$  up to isomorphism. This leads to the following

**Question 1.4.** ([4, Problem 5.4.7]): What is  $\text{Aut}(\mathbb{X})$  in other situations, and in particular in case  $M$  is a simple bimodule?

Another question is the following

**Question 1.5.** ([4, third part of Problem 5.4.6]): Find a general functorial formula for the Serre functor.

Corollary 1.2, together with results from [9], allow us to address Questions 1.3, 1.4 and 1.5 in the case that an underlying bimodule in  $\mathbf{H}$  has left-right dimension  $(2, 2)$ ,  $\text{End}(\mathcal{L})$  and  $\text{End}(\overline{\mathcal{L}})$  are commutative and isomorphic, and  $k$  is perfect with  $\text{char } k \neq 2$ . Although we are not able to provide a complete answer to Question 1.3, we are able to address a closely related problem (Proposition 4.4). To state

the result, we need to introduce some notation. If  $\sigma$  is a  $k$ -linear automorphism of  $K$ , then we let  $K_\sigma$  denote the  $k$ -central  $K - K$ -bimodule whose underlying set is  $K$  and whose bimodule structure is given by  $a \cdot x \cdot b := ax\sigma(b)$ .

**Proposition 1.6.** *Suppose  $k$  is perfect and  $\text{char } k \neq 2$ . If  $\mathbf{H}$  is a homogeneous noncommutative curve of genus zero such that an underlying bimodule,  $M$ , has left-right dimension  $(2, 2)$ , and  $\text{End}(\mathcal{L})$  and  $\text{End}(\overline{\mathcal{L}})$  are commutative and isomorphic, then the group of  $k$ -linear autoequivalences of  $\mathbf{H}$  acts transitively on line bundles if and only if there exists  $\sigma, \epsilon \in \text{Gal}(K/k)$  such that  $M \cong K_\sigma \otimes {}^*M \otimes K_\epsilon$ .*

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## 2. NONCOMMUTATIVE SYMMETRIC ALGEBRAS

The purpose of this section is to recall the definition of the noncommutative symmetric algebra of a bimodule from [13]. For the remainder of the paper, we let  $K$  and  $L$  denote finite field extensions of  $k$ . We let  $N$  denote a  $k$ -central  $K - L$ -bimodule which has finite dimension as both a  $K$ -module and an  $L$ -module. We denote the restriction of scalars of  $N$  to  $K \otimes_k 1$  (resp.  $1 \otimes_k L$ ) by  ${}_K N$  (resp.  $N_L$ ). If  $\dim_K({}_K N) = m$  and  $\dim_L(N_L) = n$ , we say  $N$  is an  $(m, n)$ -bimodule. For the remainder of the paper, all bimodules will be  $k$ -central with  $m, n < \infty$ . We will use the fact that if  $K'$  is another finite extension of  $k$ ,  $M$  is a  $K - L$ -bimodule with left basis  $\{v_1, \dots, v_m\}$  and  $N$  is an  $L - K'$ -bimodule with left basis  $\{w_1, \dots, w_n\}$ , then  $\{v_i \otimes w_j | 1 \leq i \leq m \text{ and } 1 \leq j \leq n\}$  is a left basis for  $M \otimes_L N$ .

We need to recall (from [13]) the notion of left and right dual of  $N$ . The *right dual* of  $N$ , denoted  $N^*$ , is the set  $\text{Hom}_L(N_L, L)$  with action

$$(a \cdot \psi \cdot b)(n) = a\psi(bn)$$

for all  $\psi \in \text{Hom}_L(N_L, L)$ ,  $a \in L$  and  $b \in K$ . We note that  $N^*$  is a  $L \otimes_k K$ -module.

The *left dual* of  $N$ , denoted  ${}^*N$ , is the set  $\text{Hom}_K({}_K N, K)$  with action

$$(a \cdot \phi \cdot b)(n) = b\phi(na)$$

for all  $\phi \in \text{Hom}_K({}_K N, K)$ ,  $a \in L$  and  $b \in K$ . As above,  ${}^*N$  is a  $L \otimes_k K$ -module. This assignment extends to morphisms between  $k$ -central  $K - L$ -bimodules in the obvious way.

We set

$$N^{i*} := \begin{cases} N & \text{if } i = 0, \\ (N^{i-1*})^* & \text{if } i > 0, \\ {}^*(N^{i+1*}) & \text{if } i < 0. \end{cases}$$

As in [3, Proposition 3.7], for each  $i$ , both pairs of functors

$$(- \otimes N^{i*}, - \otimes N^{i+1*})$$

and

$$(- \otimes {}^*(N^{i+1*}), - \otimes N^{i+1*})$$

between the category of  $K$ -modules and the category of  $L$ -modules, have adjoint structures. The Eilenberg-Watts theorem implies that the units of these adjoint

pairs induce maps of bimodules, and to describe them we introduce some notation. We let

$$F_i = \begin{cases} K & \text{if } i \text{ is even, and} \\ L & \text{if } i \text{ is odd.} \end{cases}$$

We suppose  $\{\phi_1, \dots, \phi_n\}$  is a right basis for  $N^{i*}$  and  $\{f_1, \dots, f_n\}$  is the corresponding dual left basis for  $N^{i+1*}$ . We suppose  $\{\phi'_1, \dots, \phi'_n\}$  is the right basis for  ${}^*(N^{i+1*})$  dual to  $\{f_1, \dots, f_n\}$ .

The unit of the first adjoint pair induces  $\eta_{i,N} : F_i \longrightarrow N^{i*} \otimes N^{i+1*}$ , defined by

$$\eta_{i,N}(a) = a \sum_j \phi_j \otimes f_j$$

while the unit of the second adjoint pair induces  $\eta'_{i,N} : F_i \longrightarrow {}^*(N^{i+1*}) \otimes N^{i+1*}$  defined by

$$\eta'_{i,N}(a) = a \sum_j \phi'_j \otimes f_j.$$

We denote the image of  $\eta_{i,N}$  by  $Q_{i,N}$  and the image of  $\eta'_{i,N}$  by  $Q'_{i,N}$ . The subscript  $N$  will be dropped when there is no chance for confusion.

The next result follows immediately from [8, Lemma 6.6].

**Lemma 2.1.** *Let  $N$  and  $P$  be  $k$ -central  $K - L$ -bimodules.*

- (1) *Given an isomorphism  $\Gamma : N^* \longrightarrow P^*$ , there exists a unique isomorphism  $\Delta : N \longrightarrow {}^*(P^*)$  such that  $(\Delta \otimes \Gamma)(Q_{0,N}) = Q'_{0,P}$ .*
- (2) *Given an isomorphism  $\Delta : {}^*N \longrightarrow {}^*P$  (resp.  $\Gamma : N \longrightarrow P$ ), there exists a unique isomorphism  $\Gamma : N \longrightarrow P$  (resp.  $\Delta : {}^*N \longrightarrow {}^*P$ ) such that  $(\Delta \otimes \Gamma)(Q_{-1,N}) = Q_{-1,P}$ .*

We now introduce a convention that will be in effect throughout this paper: all unadorned tensor products will be bimodule tensor products over the appropriate base ring.

We next recall the definition of  $\mathbb{Z}$ -algebra from [14, Section 2]: a  $\mathbb{Z}$ -algebra is a ring  $A$  with decomposition  $A = \bigoplus_{i,j \in \mathbb{Z}} A_{ij}$  into  $k$ -vector spaces, such that multiplication has the property  $A_{ij}A_{jk} \subset A_{ik}$  while  $A_{ij}A_{kl} = 0$  if  $j \neq k$ . Furthermore, for  $i \in \mathbb{Z}$ , there is a local unit  $e_i \in A_{ii}$ , such that if  $a \in A_{ij}$ , then  $e_i a = a = a e_j$ .

Now we are finally ready to recall (from [13]) the definition of the noncommutative symmetric algebra of  $N$ . All tensor products will be defined either over  $K$  or  $L$ , and the choice will be clear from context. The *noncommutative symmetric algebra generated by  $N$* , denoted  $\mathbb{S}^{n.c.}(N)$ , is the  $\mathbb{Z}$ -algebra  $\bigoplus_{i,j \in \mathbb{Z}} A_{ij}$  with components defined as follows:

- $A_{ij} = 0$  if  $i > j$ .
- $A_{ii} = K$  for  $i$  even,
- $A_{ii} = L$  for  $i$  odd, and
- $A_{ii+1} = N^{i*}$ .

In order to define  $A_{ij}$  for  $j > i + 1$ , we introduce some notation: we define  $T_{ii+1} := A_{ii+1}$ , and, for  $j > i + 1$ , we define

$$T_{ij} := A_{ii+1} \otimes A_{i+1i+2} \otimes \cdots \otimes A_{j-1j}.$$

We let  $R_{ii+1} := 0$ ,  $R_{ii+2} := Q_i$ ,

$$R_{ii+3} := Q_i \otimes N^{(i+2)*} + N^{i*} \otimes Q_{i+1},$$

and, for  $j > i + 3$ , we let

$$R_{ij} := Q_i \otimes T_{i+2j} + T_{ii+1} \otimes Q_{i+1} \otimes T_{i+3j} + \cdots + T_{ij-2} \otimes Q_{j-2}.$$

- For  $j > i + 1$ , we define  $A_{ij}$  as the quotient  $T_{ij}/R_{ij}$ .

Multiplication in  $\mathbb{S}^{n.c.}(N)$  is defined as follows:

- if  $x \in A_{ij}$ ,  $y \in A_{lk}$  and  $j \neq l$ , then  $xy = 0$ ,
- if  $x \in A_{ij}$  and  $y \in A_{jk}$ , with either  $i = j$  or  $j = k$ , then  $xy$  is induced by the usual scalar action,
- otherwise, if  $i < j < k$ , we have

$$\begin{aligned} A_{ij} \otimes A_{jk} &= \frac{T_{ij}}{R_{ij}} \otimes \frac{T_{jk}}{R_{jk}} \\ &\cong \frac{T_{ik}}{R_{ij} \otimes T_{jk} + T_{ij} \otimes R_{jk}}. \end{aligned}$$

Since  $R_{ij} \otimes T_{jk} + T_{ij} \otimes R_{jk}$  is a submodule of  $R_{ik}$ , there is thus an epi  $\mu_{ijk} : A_{ij} \otimes A_{jk} \longrightarrow A_{ik}$ .

### 3. PROOF OF THE MAIN THEOREM

Our goal in this section is to prove Theorem 1.1. We begin the section with a description of the notation we will utilize as well as a statement of our assumptions. Throughout the rest of this paper,  $\mathbf{H}$  will denote a homogeneous noncommutative curve of genus zero, and all unadorned Hom's will be over  $\mathbf{H}$ .

A *bundle* in  $\mathbf{H}$  is an object that doesn't have a simple subobject. A *line bundle* in  $\mathbf{H}$  is bundle of rank one (see [7, p.136] for the definition of rank).

- We let  $\mathcal{L}$  denote a line bundle in  $\mathbf{H}$ . There is an indecomposable bundle  $\overline{\mathcal{L}}$  and an irreducible morphism  $\mathcal{L} \longrightarrow \overline{\mathcal{L}}$  coming from the AR sequence starting at  $\mathcal{L}$  [4, 1.1.2].
- We let  $\tau^{-1}$  denote a fixed quasi-inverse of the Auslander-Reiten translation,  $\tau$ , of  $\mathbf{H}$ .
- We let  $M$  denote the  $\text{End}(\overline{\mathcal{L}}) - \text{End}(\mathcal{L})$ -bimodule  $\text{Hom}(\mathcal{L}, \overline{\mathcal{L}})$ .
- We assume that  $\text{End}(\overline{\mathcal{L}})$  and  $\text{End}(\mathcal{L})$  are commutative.

We recall that  $\text{End}(\mathcal{L})$  and  $\text{End}(\overline{\mathcal{L}})$  are automatically division rings of finite dimension over  $k$  by [7, Lemma 1.3]. Therefore, by the above assumption, they are finite extension fields of  $k$ .

If  $\mathcal{N}$  is an indecomposable bundle, we let

$$(3-1) \quad 0 \longrightarrow \mathcal{N} \xrightarrow{h} \mathcal{E} \xrightarrow{p} \tau^{-1}\mathcal{N} \longrightarrow 0.$$

be an AR sequence starting from  $\mathcal{N}$ . One can show, using results from [4, Section 1.1.2], that the only indecomposable bundles in  $\mathbf{H}$  are of the form  $\tau^i \mathcal{L}$  and  $\tau^i \overline{\mathcal{L}}$ , and  $\mathcal{E}$  is given according to the following table, where  $(m, n)$  denotes the left-right dimension of  $M$ :

$(m, n)$	$\mathcal{N}$	$\mathcal{E}$
$(2, 2)$	$\tau^i \mathcal{L}$	$\tau^i \overline{\mathcal{L}}^{\oplus 2}$
$(2, 2)$	$\tau^i \overline{\mathcal{L}}$	$\tau^{i-1} \mathcal{L}^{\oplus 2}$
$(1, 4)$	$\tau^i \mathcal{L}$	$\tau^i \overline{\mathcal{L}}$
$(1, 4)$	$\tau^i \overline{\mathcal{L}}$	$\tau^{i-1} \mathcal{L}^{\oplus 4}$

It is not hard to show, using the factorization property of AR sequences (see [2, Proposition 2.1]), that if we write  $\mathcal{E} = \mathcal{P}^{\oplus n}$  where  $\mathcal{P}$  is indecomposable, then the components of the map  $h$  in (3-1) are a left basis for  $\text{Hom}(\mathcal{N}, \mathcal{P})$ .

We will also need the fact that, according to [4, Section 1.1.2],

$$(3-2) \quad \text{Hom}(\mathcal{L}, \mathcal{N}) = 0$$

if  $\mathcal{N}$  is either  $\tau^i \mathcal{L}$  or  $\tau^i \overline{\mathcal{L}}$  and  $i > 0$ . Similarly,

$$(3-3) \quad \text{Hom}(\overline{\mathcal{L}}, \mathcal{N}) = 0$$

if  $\mathcal{N}$  is either  $\tau^i \mathcal{L}$  and  $i \geq 0$  or  $\tau^i \overline{\mathcal{L}}$  and  $i > 0$ .

Let  $\mathcal{N}$  be an indecomposable bundle and let  $\mathcal{P}$  be an indecomposable summand of  $\mathcal{E}$  in (3-1). Although it is mentioned in [4] that

$$(3-4) \quad {}^* \text{Hom}(\mathcal{N}, \mathcal{P}) \cong \text{Hom}(\mathcal{P}, \tau^{-1} \mathcal{N}),$$

the  $\text{End}(\mathcal{N})$ -module structure on  $\text{Hom}(\mathcal{P}, \tau^{-1} \mathcal{N})$  is not explicitly described. Since it will be convenient for us to describe this structure, we reconstruct (3-4) in Proposition 3.2. The proof of Proposition 3.2 is an adaptation of [2, Proposition 2.1] to our setting. We will need the following

**Lemma 3.1.** *Suppose  $\mathcal{N}$  is an indecomposable object in  $\mathcal{H}$ . If  $a \in \text{End}(\mathcal{N})$ , then there exists a unique  $f_a \in \text{End}(\mathcal{E})$  such that the diagram*

$$\begin{array}{ccc} \mathcal{N} & \xrightarrow{h} & \mathcal{E} \\ a \downarrow & & \downarrow f_a \\ \mathcal{N} & \xrightarrow[h]{} & \mathcal{E} \end{array}$$

whose horizontals are from (3-1), commutes. In addition, the function  $a \mapsto f_a$  is a  $k$ -algebra homomorphism.

*Proof.* We write  $\mathcal{E} = \mathcal{P}^{\oplus n}$ , where  $\mathcal{P}$  is an indecomposable bundle, and we use the fact that the components of  $h = (f_1, \dots, f_n)$  are a left basis for  $\text{Hom}(\mathcal{N}, \mathcal{P})$ . Therefore, if  $a \in \text{End}(\mathcal{N})$ , there exist  $a_{ij}$  in  $\text{End}(\mathcal{P})$  such that

$$(3-5) \quad f_i a = \sum_j a_{ij} f_j.$$

We let  $f_a \in \text{End}(\mathcal{P}^{\oplus n})$  denote the morphism uniquely determined by the fact that if  $g_i : \mathcal{P} \rightarrow \mathcal{P}^{\oplus n}$  denotes the  $i$ th inclusion, then  $f_a g_i = (a_{1i}, \dots, a_{ni})$ . Uniqueness of  $f_a$  follows from the fact that  $\{f_1, \dots, f_n\}$  is a left basis for  $\text{Hom}(\mathcal{N}, \mathcal{P})$ .

The proof of the fact that the function  $a \mapsto f_a$  is a  $k$ -algebra homomorphism is routine and omitted.  $\square$

**Proposition 3.2.** *Let  $\mathcal{P}$  be an indecomposable summand of  $\mathcal{E}$  in (3-1). There exists a  $k$ -algebra homomorphism*

$$\Phi : \text{End}(\mathcal{N}) \rightarrow \text{End}(\tau^{-1} \mathcal{N})$$

endowing  $\text{Hom}(\mathcal{P}, \tau^{-1} \mathcal{N})$  with an  $\text{End}(\mathcal{N}) - \text{End}(\mathcal{P})$ -bimodule structure. With this structure, there is an isomorphism of  $\text{End}(\mathcal{N}) - \text{End}(\mathcal{P})$ -bimodules

$$\Psi : {}^* \text{Hom}(\mathcal{N}, \mathcal{P}) \rightarrow \text{Hom}(\mathcal{P}, \tau^{-1} \mathcal{N}).$$

Furthermore,  $\Phi$  is induced by  $\tau^{-1}$ .

*Proof.* We define the  $k$ -algebra homomorphism  $\Phi : \text{End}(\mathcal{N}) \longrightarrow \text{End}(\tau^{-1}\mathcal{N})$  as follows: given  $a \in \text{End}(\mathcal{N})$ , we get a unique  $f_a \in \text{End}(\mathcal{E})$  such that the diagram

$$\begin{array}{ccc} \mathcal{N} & \xrightarrow{h} & \mathcal{E} \\ a \downarrow & & \downarrow f_a \\ \mathcal{N} & \xrightarrow[h]{} & \mathcal{E} \end{array}$$

commutes by Lemma 3.1. It follows that there is a unique  $g_a \in \text{End}(\tau^{-1}\mathcal{N})$  making

$$(3-6) \quad \begin{array}{ccc} \mathcal{E} & \xrightarrow{p} & \tau^{-1}\mathcal{N} \\ f_a \downarrow & & \downarrow g_a \\ \mathcal{E} & \xrightarrow{p} & \tau^{-1}\mathcal{N} \end{array}$$

commute. We define  $\Phi(a) := g_a$ . The proof that  $\Phi(a)$  is a  $k$ -algebra homomorphism is routine and omitted. Finally, the fact that  $g_a = \tau^{-1}(a)$  is an easy application of [5, Corollary 4.2] and we omit the details.

Next, we construct  $\Psi : {}^*\text{Hom}(\mathcal{N}, \mathcal{P}) \longrightarrow \text{Hom}(\mathcal{P}, \tau^{-1}\mathcal{N})$ . To this end, suppose  $\{\phi_1, \dots, \phi_n\}$  is the right basis for  ${}^*\text{Hom}(\mathcal{N}, \mathcal{P})$  dual to a left basis  $\{f_1, \dots, f_n\}$  of  $\text{Hom}(\mathcal{N}, \mathcal{P})$ . We define  $\Psi$  by letting it send  $\phi_m$  to  $p \circ g_m$  where  $g_m$  is inclusion of the  $m$ th factor of  $\mathcal{P}$  in  $\mathcal{E}$ , and we extend right-linearly.

We need to show  $\Psi$  is one-to-one, onto, and compatible with left multiplication. We show first that  $\Psi$  is one-to-one. Suppose  $a_1, \dots, a_n \in \text{End}(\mathcal{P})$  are such that

$$p\left(\sum_i g_i a_i\right) = 0.$$

Then  $\sum_i g_i a_i : \mathcal{P} \longrightarrow \mathcal{E}$  factors through the kernel of  $p$ . Since  $\text{Hom}(\mathcal{P}, \mathcal{N}) = 0$  by (3-2), it follows that  $a_1 = \dots = a_n = 0$ . Therefore,  $\Psi$  is one-to-one. The fact that  $\Psi$  is onto will follow from the fact that the right dimension of  $\text{Hom}(\mathcal{P}, \tau^{-1}\mathcal{N})$  is equal to  $n$ . To prove this, we consider the long exact sequence resulting from applying the functor  $\text{Hom}(\mathcal{P}, -)$  to the short exact sequence (3-1), and note that  $0 = D \text{Hom}(\mathcal{N}, \tau\mathcal{P}) \cong \text{Ext}^1(\mathcal{P}, \mathcal{N})$  by (3-2) and (3-3).

We now show that  $\Psi$  is compatible with the  $\text{End}(\mathcal{N})$ -object structure. To this end, we suppose the right action on the left basis  $\{f_1, \dots, f_n\}$  is given by (3-5). Then, by [3, Lemma 3.4], the left action on the right basis  $\{\phi_1, \dots, \phi_n\}$  of  ${}^*\text{Hom}(\mathcal{N}, \mathcal{P})$  is given by

$$a\phi_i = \sum_j \phi_j a_{ji}.$$

Therefore, by definition of  $\Psi$  we have

$$\begin{aligned} \Psi(a \cdot (\sum_i \phi_i b_i)) &= \Psi(\sum_{i,j} \phi_j a_{ji} b_i) \\ &= p(\sum_j g_j (\sum_i a_{ji} b_i)). \end{aligned}$$

On the other hand, if  $b_1, \dots, b_n \in \text{End}(\mathcal{P})$ , then by definition of  $f_a$  from Lemma 3.1 we have

$$(3-7) \quad f_a(b_1, \dots, b_n) = (\sum_i a_{1i} b_i, \dots, \sum_i a_{ni} b_i).$$



Therefore,

$$\begin{aligned} a \cdot \left( \sum_i p g_i b_i \right) &= p(f_a(\sum_i g_i b_i)) \\ &= p\left(\sum_j g_j \left(\sum_i a_{ji} b_i\right)\right). \end{aligned}$$

as desired.  $\square$

For the remainder of this section,  $\Phi$  and  $\Psi$  will refer to the maps defined in Proposition 3.2.

The following is an adaptation of [2, Proposition 2.2].

**Lemma 3.3.** *Let  $\mathcal{P}$  be an indecomposable summand of  $\mathcal{E}$  in (3-1). Let  $\{f_1, \dots, f_n\}$  denote a left basis for  $\text{Hom}(\mathcal{N}, \mathcal{P})$ , and let  $\{\phi_1, \dots, \phi_n\}$  denote the corresponding right dual basis for  ${}^*\text{Hom}(\mathcal{N}, \mathcal{P})$ . Then, under the composition*

$$\begin{aligned} {}^*\text{Hom}(\mathcal{N}, \mathcal{P}) \otimes_{\text{End}(\mathcal{P})} \text{Hom}(\mathcal{N}, \mathcal{P}) &\xrightarrow{\Psi \otimes 1} \text{Hom}(\mathcal{P}, \tau^{-1}\mathcal{N}) \otimes_{\text{End}(\mathcal{P})} \text{Hom}(\mathcal{N}, \mathcal{P}) \\ &\longrightarrow \text{Hom}(\mathcal{N}, \tau^{-1}\mathcal{N}) \end{aligned}$$

whose second arrow is induced by composition, the element  $\sum_i \phi_i \otimes f_i$  goes to zero.

*Proof.* We let  $g_i : \mathcal{P} \rightarrow \mathcal{P}^{\oplus n}$  denote the  $i$ th inclusion. By definition of  $\Psi$ , the element  $\sum_i \phi_i \otimes f_i$  maps to  $p(\sum_i g_i f_i)$ . But this is zero as it is the composition of maps in (3-1).  $\square$

For the next result, recall the definition of  $Q_i$  from Section 2.

**Proposition 3.4.** *For  $i \in \mathbb{Z}$  even, there is a canonical isomorphism of  $\text{End}(\overline{\mathcal{L}}) - \text{End}(\mathcal{L})$ -bimodules*

$$\Psi_i : M^{i*} \longrightarrow \text{Hom}(\tau^{\frac{i}{2}}\mathcal{L}, \tau^{\frac{i}{2}}\overline{\mathcal{L}}),$$

and for  $i \in \mathbb{Z}$  odd there is a canonical isomorphism of  $\text{End}(\mathcal{L}) - \text{End}(\overline{\mathcal{L}})$ -bimodules

$$\Psi_i : M^{i*} \longrightarrow \text{Hom}(\tau^{\frac{i+1}{2}}\overline{\mathcal{L}}, \tau^{\frac{i-1}{2}}\mathcal{L}),$$

where the bimodule structure on the codomain is induced by the appropriate power of  $\tau$ . Furthermore, for  $i \in \mathbb{Z}$ ,  $Q_i \subset \ker(\Psi_i \otimes \Psi_{i+1})$ .

*Proof.* First we construct  $\Psi_i$  for  $i \geq 0$ . We proceed by induction on  $i$ . If  $i = 0$ , the isomorphism is equality.

Now, suppose  $i + 1$  is odd. Then there is a composition,  $\Psi'_i$  of isomorphisms

$$\begin{aligned} {}^*((M^{i*})^*) &\xrightarrow{\cong} M^{i*} \\ &\xrightarrow{\Psi_i} \text{Hom}(\tau^{\frac{i}{2}}\mathcal{L}, \tau^{\frac{i}{2}}\overline{\mathcal{L}}) \\ &\xrightarrow{\cong} \text{Hom}(\tau^{\frac{i}{2}}\mathcal{L}, \tau^{-1}\tau^{\frac{i+2}{2}}\overline{\mathcal{L}}) \\ &\xrightarrow{\cong} {}^*\text{Hom}(\tau^{\frac{i+2}{2}}\overline{\mathcal{L}}, \tau^{\frac{i}{2}}\mathcal{L}) \end{aligned}$$

as follows: the first isomorphism is from Lemma 2.1(1) with  $\Gamma = \text{id}_{M^{i+1*}}$ , the third isomorphism is induced by a natural transformation

$$(3-8) \quad \tau^{\frac{i}{2}} \longrightarrow \tau^{-1}\tau^{\frac{i+2}{2}},$$

and the fourth isomorphism is an application of Proposition 3.2 with  $\mathcal{N} = \tau^{\frac{i+2}{2}}\overline{\mathcal{L}}$  and with  $\mathcal{P} = \tau^{\frac{i}{2}}\mathcal{L}$ .

We claim that these isomorphisms are all compatible with the  $\text{End}(\overline{\mathcal{L}}) - \text{End}(\mathcal{L})$ -bimodule structures. For, by the naturality of (3-8), the third isomorphism is compatible with the  $\text{End}(\overline{\mathcal{L}}) - \text{End}(\mathcal{L})$ -bimodule structures. By Proposition 3.2, the fourth isomorphism is as  $\text{End}(\tau^{\frac{i+2}{2}}\overline{\mathcal{L}}) - \text{End}(\tau^{\frac{i}{2}}\mathcal{L})$ -bimodules. Hence, under the isomorphisms induced by the appropriate powers of  $\tau$ , this is an isomorphism of  $\text{End}(\overline{\mathcal{L}}) - \text{End}(\mathcal{L})$ -bimodules.

It follows from Lemma 2.1(2) that if  $\{f_1, \dots, f_n\}$  is a left basis for the bimodule  $\text{Hom}(\tau^{\frac{i+2}{2}}\overline{\mathcal{L}}, \tau^{\frac{i}{2}}\mathcal{L})$  and  $\{\phi_1, \dots, \phi_n\}$  is the corresponding right dual basis for  ${}^*\text{Hom}(\tau^{\frac{i+2}{2}}\overline{\mathcal{L}}, \tau^{\frac{i}{2}}\mathcal{L})$ , then there exists a unique isomorphism of bimodules

$$\Psi_{i+1} : M^{i+1*} \longrightarrow \text{Hom}(\tau^{\frac{i+2}{2}}\overline{\mathcal{L}}, \tau^{\frac{i}{2}}\mathcal{L})$$

such that  $(\Psi'_i) \otimes \Psi_{i+1}$  maps  $Q'_{i,M}$  to the submodule generated by  $\sum_i \phi_i \otimes f_i$ .

A similar construction yields  $\Psi_{i+1}$  for  $i+1 > 0$  even, and we omit the details.

Next, we construct  $\Psi_i$  in case  $i \leq 0$  inductively. Suppose  $i-1$  is odd, and suppose we have an isomorphism  $\Psi_i : M^{i*} \longrightarrow \text{Hom}(\tau^{\frac{i}{2}}\mathcal{L}, \tau^{\frac{i}{2}}\overline{\mathcal{L}})$ . Then we have a composition of isomorphisms

$$\begin{aligned} M^{i-1*} &\xrightarrow{=} {}^*(M^{i*}) \\ &\xrightarrow{\Psi'_{i-1}} {}^*\text{Hom}(\tau^{\frac{i}{2}}\mathcal{L}, \tau^{\frac{i}{2}}\overline{\mathcal{L}}) \\ &\xrightarrow{\cong} \text{Hom}(\tau^{\frac{i}{2}}\overline{\mathcal{L}}, \tau^{-1}\tau^{\frac{i}{2}}\mathcal{L}) \\ &\xrightarrow{\cong} \text{Hom}(\tau^{\frac{i}{2}}\mathcal{L}, \tau^{\frac{i-2}{2}}\overline{\mathcal{L}}) \end{aligned}$$

whose second arrow  $\Psi'_{i-1}$  is the isomorphism  $\Delta$  from Lemma 2.1(2) with  $\Gamma = \Psi_i$  and whose other arrows are defined as in the  $i > 0$  case. As above, this isomorphism is compatible with the  $\text{End}(\mathcal{L}) - \text{End}(\overline{\mathcal{L}})$ -bimodule structures.

The construction of  $\Psi_{i-1}$  in case  $i \leq 0$ ,  $i-1$  even is similar and we leave the details to the reader.

Now we prove the second part of the proposition. We do this in case  $i$  is even and  $i \geq 0$ . The proofs in other cases are similar and omitted. By the definition of  $\Psi'_i$  and  $\Psi_{i+1}$  above, and by Lemma 3.3, we know that  $Q'_{i,M}$  maps to zero under the composition

$$\begin{aligned} {}^*(M^{i+1*}) \otimes M^{i+1*} &\xrightarrow{\Psi'_i \otimes \Psi_{i+1}} {}^*\text{Hom}(\tau^{\frac{i+2}{2}}\overline{\mathcal{L}}, \tau^{\frac{i}{2}}\mathcal{L}) \otimes \text{Hom}(\tau^{\frac{i+2}{2}}\overline{\mathcal{L}}, \tau^{\frac{i}{2}}\mathcal{L}) \\ &\xrightarrow{\cong} \text{Hom}(\tau^{\frac{i}{2}}\mathcal{L}, \tau^{-1}\tau^{\frac{i+2}{2}}\overline{\mathcal{L}}) \otimes \text{Hom}(\tau^{\frac{i+2}{2}}\overline{\mathcal{L}}, \tau^{\frac{i}{2}}\mathcal{L}) \\ &\longrightarrow \text{Hom}(\tau^{\frac{i+2}{2}}\overline{\mathcal{L}}, \tau^{-1}\tau^{\frac{i+2}{2}}\overline{\mathcal{L}}) \end{aligned}$$

whose second arrow is from Proposition 3.2 and whose last arrow is composition. Therefore, by definition of  $\Psi_i$ ,  $Q'_{i,M}$  maps to zero under the composition

$$\begin{aligned} {}^*(M^{i+1*}) \otimes M^{i+1*} &\xrightarrow{\cong} M^{i*} \otimes M^{i+1*} \\ &\xrightarrow{\Psi_i \otimes \Psi_{i+1}} \text{Hom}(\tau^{\frac{i}{2}}\mathcal{L}, \tau^{\frac{i}{2}}\overline{\mathcal{L}}) \otimes \text{Hom}(\tau^{\frac{i+2}{2}}\overline{\mathcal{L}}, \tau^{\frac{i}{2}}\mathcal{L}) \\ &\longrightarrow \text{Hom}(\tau^{\frac{i+2}{2}}\overline{\mathcal{L}}, \tau^{\frac{i}{2}}\overline{\mathcal{L}}) \end{aligned}$$

whose first arrow is from Lemma 2.1(1) with  $\Gamma = \text{id}_{M^{i+1*}}$  and whose last arrow is composition. But this implies, by Lemma 2.1(1), that  $Q_{i,M}$  goes to zero under the composition of the last three maps above, completing the proof in this case.  $\square$

**Lemma 3.5.** *Let  $\mathcal{F}$  and  $\mathcal{G}$  be indecomposable bundles in  $\mathbf{H}$ , and let  $\text{rdim}(\text{Hom}(\mathcal{F}, \mathcal{G}))$  denote the dimension of  $\text{Hom}(\mathcal{F}, \mathcal{G})$  as a  $\text{End}(\mathcal{F})$ -vector space. If  $i \geq 0$  is an integer then we have*

$\mathcal{F}$	$\mathcal{G}$	$\text{rdim}(\text{Hom}(\mathcal{F}, \mathcal{G}))$ in $(1, 4)$ case	$\text{rdim}(\text{Hom}(\mathcal{F}, \mathcal{G}))$ in $(2, 2)$ case
$\mathcal{L}$	$\tau^{-i}\mathcal{L}$	$2i + 1$	$2i + 1$
$\overline{\mathcal{L}}$	$\tau^{-i}\overline{\mathcal{L}}$	$i$	$2i$
$\overline{\mathcal{L}}$	$\tau^{-i}\mathcal{L}$	$2i + 1$	$2i + 1$
$\mathcal{L}$	$\tau^{-i}\overline{\mathcal{L}}$	$4i + 4$	$2i + 2$

If  $i < 0$  all of the above Hom spaces are zero.

*Proof.* We prove the formulas in case  $M$  is a  $(1, 4)$ -bimodule. The proof in the case that  $M$  is a  $(2, 2)$ -bimodule is similar and omitted. The first formula is [4, Section 1.1.15]. We prove the second and third formulas by induction on  $i$ . The  $i = 0$  case follows from (3-3). Now suppose  $i \geq 0$  and consider the AR sequences

$$(3-9) \quad 0 \longrightarrow \tau^{-i}\mathcal{L} \longrightarrow \tau^{-i}\overline{\mathcal{L}} \longrightarrow \tau^{-(i+1)}\mathcal{L} \longrightarrow 0$$

and

$$(3-10) \quad 0 \longrightarrow \tau^{-i}\overline{\mathcal{L}} \longrightarrow \tau^{-(i+1)}\mathcal{L}^{\oplus 4} \longrightarrow \tau^{-(i+1)}\overline{\mathcal{L}} \longrightarrow 0.$$

Applying  $\text{Hom}(\overline{\mathcal{L}}, -)$  to (3-9) yields a short exact sequence

$$0 \longrightarrow \text{Hom}(\overline{\mathcal{L}}, \tau^{-i}\mathcal{L}) \longrightarrow \text{Hom}(\overline{\mathcal{L}}, \tau^{-i}\overline{\mathcal{L}}) \longrightarrow \text{Hom}(\overline{\mathcal{L}}, \tau^{-(i+1)}\mathcal{L}) \longrightarrow 0$$

by (3-2), and the second formula in the  $i + 1$  case follows from the induction hypothesis applied to this sequence.

The third formula in the  $i + 1$  case now follows by applying  $\text{Hom}(\overline{\mathcal{L}}, -)$  to (3-10) and using (3-2), then employing the  $i + 1$  case of the second formula and the induction hypothesis. Now the second and third formulas follow by induction.

Finally, the fourth formula follows in a similar fashion by applying  $\text{Hom}(\mathcal{L}, -)$  to (3-9) and using the first formula.

The last statement follows from (3-2) and (3-3).  $\square$

We now define a sequence of objects in  $\mathbf{H}$  which will be used to construct a  $\mathbb{Z}$ -algebra coordinate ring for  $\mathbf{H}$ . For  $n \in \mathbb{Z}$ , we define

$$(3-11) \quad \mathcal{O}(n) := \begin{cases} \tau^{\frac{-n}{2}}\overline{\mathcal{L}} & \text{if } n \text{ is even} \\ \tau^{\frac{-(n+1)}{2}}\mathcal{L} & \text{if } n \text{ is odd.} \end{cases}$$

We define a  $\mathbb{Z}$ -algebra  $H$  by setting

$$H_{ij} = \begin{cases} \text{Hom}(\mathcal{O}(-j), \mathcal{O}(-i)) & \text{if } j \geq i \\ 0 & \text{if } i > j \end{cases}$$

and defining multiplication as composition.

The next result follows immediately from Lemma 3.5.

**Corollary 3.6.** *Suppose  $j \geq i$ . If  $M$  is a  $(1, 4)$ -bimodule, then the right dimension of  $H_{ij}$  is*

$$\begin{cases} j - i + 1 & \text{if } i \text{ and } j \text{ have the same parity,} \\ \frac{j-i+1}{2} & \text{if } i \text{ is odd and } j \text{ is even, and} \\ 2j - 2i + 2 & \text{if } i \text{ is even and } j \text{ is odd.} \end{cases}$$

If  $M$  is a  $(2, 2)$ -bimodule, then the right dimension of  $H_{ij}$  is  $j - i + 1$ .

In order to prove Proposition 3.8, we will need the following technical lemma.

**Lemma 3.7.** *Let  $M$  be a  $(1, 4)$ -bimodule, and write  $\mathbb{S}^{n.c.}(M)_{ij} = T_{ij}/R_{ij}$ , where  $T$  and  $R$  are defined in Section 2. Suppose  $i, j \in \mathbb{Z}$  are such that  $j$  is odd and  $j > i$ . Finally, suppose  $v \in T_{ij}$  has the property that there exist two right-independent vectors  $g_1, g_2 \in T_{j+1}$  such that  $v \otimes g_l \in R_{ij+1}$  for  $l = 1, 2$ . Then  $v \in R_{ij}$ .*

*Proof.* We proceed by induction on  $j - i$ . Suppose  $j - i = 1$ , so that  $v \in M^{j*}$ . Let  $\phi$  be a nonzero element of  $M^{j*}$  so that  $\phi$  is a right basis for  $M^{j*}$ , and let  $\phi^* \in M^{j+1*}$  denote the corresponding right dual. Then, for  $l = 1, 2$ , there exist  $a, b_l \in \text{End}(\overline{\mathcal{L}})$  and  $c_l \in \text{End}(\mathcal{L})$  such that  $v = \phi a$ ,

$$g_l = b_l \phi^*,$$

and

$$\phi a \otimes b_l \phi^* = \phi \otimes \phi^* c_l.$$

Therefore, for  $l = 1, 2$ ,  $ab_l \phi^* = \phi^* c_l$ . Thus, if  $a \neq 0$ , then  $c_l \neq 0$ , and so for  $l = 1, 2$ ,  $g_l c_l^{-1} = a^{-1} \phi^*$ . This contradicts the right independence of  $g_1, g_2$ . It follows that  $a = 0$  and hence  $v = 0$ , so that  $v \in R_{ii+1}$ .

Now suppose  $j - i > 1$  and retain the notation in the previous paragraph. Since, for  $l = 1, 2$ ,

$$v \otimes g_l \in R_{ij+1} = T_{ij-1} \otimes Q_j + R_{ij} \otimes M^{j+1*},$$

there exist  $h_l \in T_{ij-1}$  such that

$$v \otimes g_l - h_l \otimes \phi \otimes \phi^* \in R_{ij} \otimes M^{j+1*}.$$

Since  $\phi^*$  is a left basis for  $M^{j+1*}$ , it follows that, for  $l = 1, 2$ ,

$$(3-12) \quad vb_l - h_l \otimes \phi \in R_{ij},$$

and hence

$$(3-13) \quad h_1 \otimes \phi b_1^{-1} - h_2 \otimes \phi b_2^{-1} \in R_{ij}.$$

It is straightforward to check that the set  $\{f_1 := \phi b_1^{-1}, f_2 := \phi b_2^{-1}\}$  is left independent so that it can be extended to a left basis  $\{f_n\}_{n=1}^4$  of  $M^{j*}$ . If  $j = i + 2$ , then (3-13) implies that  $h_1 = 0$  so that (3-12) implies that  $v \in R_{ij}$ . If  $j > i + 2$ , then (3-13) implies there is an  $h \in T_{ij-2}$  such that

$$h_1 \otimes f_1 - h_2 \otimes f_2 - h \otimes \sum_l^* f_l \otimes f_l \in R_{ij-1} \otimes M^{j*}$$

where  $\{^* f_n\}$  is the dual basis to  $\{f_n\}$ . Therefore,  $h \otimes ^* f_3, h \otimes ^* f_4$  and  $h_1 - h \otimes ^* f_1$  are in  $R_{ij-1}$ . By induction,  $h \in R_{ij-2}$ , so that  $h_1 \in R_{ij-1}$ . Therefore, by (3-12),  $v \in R_{ij}$ .  $\square$

**Proposition 3.8.** *The right dimension of  $H_{ij}$  equals that of  $\mathbb{S}^{n.c.}(M)_{ij}$  for all integers  $i, j$ .*

*Proof.* If  $M$  is a  $(2, 2)$ -bimodule, the result follows from Corollary 3.6 and [13, Theorem 6.1.2(1)].

Now, suppose that  $M$  is a  $(1, 4)$ -bimodule. In this case, we prove that, as in [13, Theorem 6.1.2 (2)], the exact sequence induced by multiplication

$$(3-14) \quad \mathbb{S}^{n.c.}(M)_{ij-1} \otimes Q_j \rightarrow \mathbb{S}^{n.c.}(M)_{ij} \otimes M^{j+1*} \rightarrow \mathbb{S}^{n.c.}(M)_{i,j+1} \rightarrow 0$$

is exact on the left for all  $i \in \mathbb{Z}$ . The result will follow from this and induction in light of Corollary 3.6, as the reader can check.

We retain the notation from the statement of Lemma 3.7. Left exactness of (3-14) is equivalent to the equality

$$(3-15) \quad R_{ij} \otimes M^{j+1*} \cap T_{ij-1} \otimes Q_j = R_{ij-1} \otimes Q_j$$

for all  $i, j \in \mathbb{Z}$ . For  $j < i + 2$ , both sides of (3-15) are zero. Now suppose  $j \geq i + 2$ . There are two cases to consider.

*Case 1:  $j$  is odd.* Let  $\phi \in M^{j*}$  be nonzero, so that it is a right basis for  $M^{j*}$ , and let  $\phi^* \in M^{j+1*}$  denote its right dual. Extend  $f_1 := \phi$  to a left basis  $\{f_1, \dots, f_4\}$  of  $M^{j*}$ , and let  $\{^*f_1, \dots, ^*f_4\}$  denote the corresponding dual right basis for  $M^{j-1*}$ .

Suppose  $v \in T_{ij-1}$  is such that  $v \otimes \phi \otimes \phi^*$  is in the left-hand side of (3-15). Now we consider two sub-cases: if  $j = i + 2$ , then there exists  $a \in \text{End}(\overline{\mathcal{L}})$  such that

$$v \otimes \phi \otimes \phi^* = a\left(\sum_l ^*f_l \otimes f_l\right) \otimes \phi^*.$$

Since  $\phi^*$  is a left basis for  $M^{j+1*}$ , we conclude that

$$v \otimes \phi = v \otimes f_1 = a\left(\sum_l ^*f_l \otimes f_l\right).$$

Therefore, since  $\{f_l\}_{l=1}^4$  is a left basis for  $M^{j*}$  we conclude that  $a$ , and hence  $v$ , equals zero.

If  $j > i + 2$ , then there exists  $w \in R_{ij-1} \otimes M^{j*}$  and  $u \in T_{ij-2}$  such that

$$v \otimes \phi \otimes \phi^* = w \otimes \phi^* + u \otimes \left(\sum_l ^*f_l \otimes f_l\right) \otimes \phi^*.$$

Therefore,

$$v \otimes f_1 - u \otimes \left(\sum_l ^*f_l \otimes f_l\right) \in R_{ij-1} \otimes M^{j*},$$

so that, since  $\{f_l\}_{l=1}^4$  is a left basis for  $M^{j*}$ , we conclude that

$$(3-16) \quad v - u \otimes ^*f_1 \in R_{ij-1}$$

and

$$(3-17) \quad u \otimes ^*f_j \in R_{ij-1}$$

for  $j \neq 1$ . It now follows from (3-17) and Lemma 3.7 that  $u \in R_{ij-2}$  so that (3-16) implies that  $v \in R_{ij-1}$  as desired, completing the proof in the case that  $j$  is odd.

*Case 2:  $j$  is even.* Let  $\phi \in M^{j*}$  be nonzero so that it forms a left basis for  $M^{j*}$ , let  $^*\phi$  be its left dual in  $M^{j-1*}$ , let  $\{f_1 := \phi, f_2, f_3, f_4\}$  denote a right basis for  $M^{j*}$ , and let  $\{f_1^*, \dots, f_4^*\}$  denote the corresponding dual left basis for  $M^{j+1*}$ . Finally, let  $a_l \in \text{End}(\overline{\mathcal{L}})$  be such that

$$a_l \phi = f_l.$$

Suppose that  $v \in T_{ij-1}$  is such that  $v \otimes \sum_l f_l \otimes f_l^*$  is an element of the left-hand side of (3-15). As in the case when  $j$  is odd, there are two sub-cases to consider. We leave the  $j = i + 2$  case as a straightforward exercise. Next, suppose  $j > i + 2$ . Arguing as in the  $j$  odd case, we conclude that, for  $1 \leq l \leq 4$ , there exist  $w_l \in T_{ij-2}$  such that

$$v \otimes a_l \phi - w_l \otimes ^*\phi \otimes \phi \in R_{ij-1} \otimes M^{j*}.$$

Since  $\phi$  is a left basis for  $M^{j*}$ , this implies that, for all  $l$ ,

$$(3-18) \quad va_l - w_l \otimes {}^*\phi \in R_{ij-1}.$$

Therefore, for  $1 \leq m \leq n \leq 4$ ,

$$(3-19) \quad w_m \otimes {}^*\phi a_m^{-1} - w_n \otimes {}^*\phi a_n^{-1} \in R_{ij-1}.$$

As one can check,  $\{g_l := {}^*\phi a_l^{-1}\}_{l=1}^4$  is left independent. It follows easily that if  $j = i + 3$ , then  $w_m = 0$ , so that (3-18) implies  $v \in R_{ij-1}$ . Thus, it remains to prove the result in case  $j > i + 3$ . To this end, if we let  $\{{}^*g_1, \dots, {}^*g_4\}$  denote the corresponding dual basis of  $\{g_1, \dots, g_4\}$ , then (3-19) implies that for  $1 \leq m \leq n \leq 4$ , there exists  $w \in M^{j-3*}$  such that

$$w_m \otimes g_m - w_n \otimes g_n - w \otimes \left( \sum_l {}^*g_l \otimes g_l \right) \in R_{ij-2} \otimes V^{j-1*}.$$

Thus, for  $m \neq n$  and  $l \neq m, n$ ,  $w \otimes {}^*g_l \in R_{ij-2}$ , so that Lemma 3.7 implies that  $w \in R_{ij-3}$ , and  $w_m \otimes g_m - w \otimes {}^*g_m \otimes g_m \in R_{ij-2} \otimes V^{j-1*}$  so that  $w_m \otimes g_m \in R_{ij-2} \otimes V^{j-1*}$ . Thus  $w_m \in R_{ij-2}$  and so (3-18) implies that  $v \in R_{ij-1}$  as desired.  $\square$

In order to state the next proposition, we need to introduce some terminology. Following [1, Section 2] and [14, Section 2], if  $A$  is a  $\mathbb{Z}$ -algebra, we let  $\mathbf{Gr}A$  denote the category of graded right  $A$ -modules and we let  $\mathbf{Tors}A$  denote the full subcategory of  $\mathbf{Gr}A$  consisting of objects which are direct limits of right bounded modules. We let  $\mathbf{Proj}A$  denote the quotient of  $\mathbf{Gr}A$  by  $\mathbf{Tors}A$ . We also need to recall from [11, Definition 2.1.4] that a sequence of objects  $\{\mathcal{P}(n)\}_{n \in \mathbb{Z}}$  in a locally noetherian category  $\mathbf{C}$  is *ample* if

- (1) for every object  $\mathcal{A} \in \mathbf{C}$ , there are positive integers  $l_1, \dots, l_p$  and an epimorphism  $\oplus_{i=1}^p \mathcal{P}(-l_i) \rightarrow \mathcal{A}$ , and
- (2) if  $\mathcal{A}, \mathcal{B}$  are objects in  $\mathbf{C}$  and  $f : \mathcal{A} \rightarrow \mathcal{B}$  is an epimorphism, then the induced map  $\mathrm{Hom}_{\mathbf{C}}(\mathcal{P}(-n), \mathcal{A}) \rightarrow \mathrm{Hom}_{\mathbf{C}}(\mathcal{P}(-n), \mathcal{B})$  is surjective for all  $n \gg 0$ .

**Proposition 3.9.** *The sequence  $\{\mathcal{O}(n)\}_{n \in \mathbb{Z}}$  defined in (3-11) is an ample sequence of noetherian objects in  $\mathbf{H}$ . Therefore,  $H$  is noetherian (i.e. the category  $\mathbf{Gr}H$  is locally noetherian) and*

$$\tilde{H} \equiv \mathbf{Proj}H.$$

*Proof.* The first part of the proposition follows from the second part by [11, Theorem 11.1.1]. We now prove the second part. We first recall that  $(\mathcal{L}, \tau^{-1})$  is an ample pair [4, p. 3], i.e., the sequence  $\{\tau^{-n}\mathcal{L}\}_{n \in \mathbb{Z}}$  is an ample sequence, so that part (1) of the definition of ampleness for  $\{\mathcal{O}(n)\}_{n \in \mathbb{Z}}$  follows immediately. It also follows from this that if

$$0 \rightarrow \mathcal{C} \rightarrow \mathcal{A} \rightarrow \mathcal{B} \rightarrow 0$$

is a short exact sequence in  $\mathbf{H}$ , then  $\mathrm{Ext}^1(\tau^n \mathcal{L}, \mathcal{C}) = 0 = \mathrm{Ext}^1(\tau^{n+1} \mathcal{L}, \mathcal{C})$  for some positive  $n$ . Therefore, by applying  $\mathrm{Hom}(-, \mathcal{C})$  to the AR sequence

$$0 \rightarrow \tau^{n+1} \mathcal{L} \rightarrow \tau^{n+1} \overline{\mathcal{L}}^{\oplus m} \rightarrow \tau^n \mathcal{L} \rightarrow 0$$

where  $m = 1$  if  $M$  is a  $(1, 4)$ -bimodule and  $m = 2$  if  $M$  is a  $(2, 2)$ -bimodule, we conclude that

$$\mathrm{Ext}^1(\tau^{n+1} \overline{\mathcal{L}}, \mathcal{C}) = 0.$$

Part (2) in the definition of ampleness for  $\{\mathcal{O}(n)\}_{n \in \mathbb{Z}}$  follows.  $\square$

**Theorem 3.10.** *Suppose  $H$  is a homogeneous noncommutative curve of genus zero with underlying bimodule  $M$  such that  $\text{End}(\mathcal{L})$  and  $\text{End}(\overline{\mathcal{L}})$  are commutative. Then there exists a  $k$ -linear isomorphism of  $\mathbb{Z}$ -algebras*

$$\mathbb{S}^{n.c.}(M) \longrightarrow H.$$

Therefore,  $\mathbb{P}^{n.c.}(M) \equiv \tilde{H}$ .

Conversely, if  $K$  and  $L$  are finite extensions of  $k$  and  $N$  is a  $k$ -central  $K - L$ -bimodule of left-right dimension  $(1, 4)$  or  $(2, 2)$ , then there exists a homogeneous noncommutative curve  $H$  with underlying bimodule  $N$  such that  $\mathbb{P}^{n.c.}(N) \equiv \tilde{H}$ .

*Proof.* We define a  $\mathbb{Z}$ -algebra isomorphism

$$\Lambda : \mathbb{S}^{n.c.}(M) \longrightarrow H$$

as follows. We let

$$\Lambda_{ij} = 0$$

for  $i > j$ . For  $i$  odd we let

$$\Lambda_{ii} : \text{End}(\mathcal{L}) \longrightarrow \text{End}(\mathcal{O}(-i)) = \text{End}(\tau^{\frac{i-1}{2}} \mathcal{L})$$

be induced by  $\tau^{\frac{i-1}{2}}$ , and for  $i$  even we let

$$\Lambda_{ii} : \text{End}(\overline{\mathcal{L}}) \longrightarrow \text{End}(\tau^{\frac{i}{2}} \overline{\mathcal{L}})$$

be induced by  $\tau^{\frac{i}{2}}$ . For  $i, j \in \mathbb{Z}$  with  $i < j$ , we let

$$\Lambda_{ij} : \mathbb{S}^{n.c.}(M)_{ij} \longrightarrow \text{Hom}(\mathcal{O}(-j), \mathcal{O}(-i))$$

denote the map induced by the composition

$$\begin{aligned} M^{i*} \otimes \cdots \otimes M^{j-1*} &\rightarrow \text{Hom}(\mathcal{O}(-i-1), \mathcal{O}(-i)) \otimes \cdots \otimes \text{Hom}(\mathcal{O}(-j), \mathcal{O}(-j+1)) \\ &\rightarrow \text{Hom}(\mathcal{O}(-j), \mathcal{O}(-i)) \end{aligned}$$

whose first arrow is from Proposition 3.4 and whose second arrow is composition. By Proposition 3.4, this map does indeed factor through  $\mathbb{S}^{n.c.}(M)_{ij}$ . To show that  $\Lambda_{ij}$  is an isomorphism in this case, we first note that the factorization property of AR sequences implies that the second map in the above composition is surjective. Therefore,  $\Lambda_{ij}$  is surjective. The fact that  $\Lambda_{ij}$  is injective now follows from Proposition 3.8.

To complete the proof of the first part of the theorem, it remains to check that  $\Lambda$  is compatible with multiplication. This straightforward exercise is left to the reader.

The second part of the theorem follows from the fact that given  $N$  as in the hypotheses, there exists an  $H$  with underlying bimodule  $N$  by [4, Section 0.5.2]. Then one applies the first part of the theorem.  $\square$

The next result follows immediately from Theorem 1.1.

**Corollary 3.11.** *A homogeneous noncommutative curve of genus zero,  $H$ , with underlying bimodule  $M$  is arithmetic if and only if  $\text{End}(\mathcal{L})$  is  $k$ -isomorphic to  $\text{End}(\overline{\mathcal{L}})$  and each is commutative. In this case,  $\mathbb{P}^{n.c.}(M) \equiv \tilde{H}$ .*

## 4. APPLICATIONS

In this section, we assume  $V$  and  $W$  are  $(2, 2)$ -bimodules over two copies of the same field  $K$ , which is of finite dimension over  $k$ . We will need to recall three types of canonical equivalences between noncommutative projective lines, studied in [9]. If  $\delta, \epsilon \in \text{Gal}(K/k)$  and we define

$$\zeta_i = \begin{cases} \delta & \text{if } i \text{ is even} \\ \epsilon & \text{if } i \text{ is odd,} \end{cases}$$

then twisting by the sequence  $\{K_{\zeta_i}\}_{i \in \mathbb{Z}}$  (see Section 1.3 for the definition of  $K_{\zeta_i}$ ) in the sense of [13, Section 3.2] induces an equivalence of categories

$$T_{\delta, \epsilon} : \mathbb{P}^{n.c.}(V) \longrightarrow \mathbb{P}^{n.c.}(K_{\delta^{-1}} \otimes_K V \otimes_K K_{\epsilon}).$$

Next, let  $\phi : V \longrightarrow W$  denote an isomorphism of  $K - K$ -bimodules. Then  $\phi$  induces an isomorphism  $\mathbb{S}^{n.c.}(V) \longrightarrow \mathbb{S}^{n.c.}(W)$ , which in turn induces an equivalence

$$\Phi : \mathbb{P}^{n.c.}(V) \longrightarrow \mathbb{P}^{n.c.}(W).$$

Finally, shifting graded modules by an integer,  $i$ , induces an equivalence

$$[i] : \mathbb{P}^{n.c.}(V) \longrightarrow \mathbb{P}^{n.c.}(V^{i*}),$$

and we have the following classification of equivalences between noncommutative projective lines ([9, Theorem 6.9]):

**Theorem 4.1.** *Suppose  $k$  is a perfect field with  $\text{char } k \neq 2$  and let  $F : \mathbb{P}^{n.c.}(V) \longrightarrow \mathbb{P}^{n.c.}(W)$  be a  $k$ -linear equivalence. Then there exists  $\delta, \epsilon \in \text{Gal}(K/k)$ , an isomorphism  $\phi : K_{\delta^{-1}} \otimes_K V \otimes_K K_{\epsilon} \longrightarrow W^{i*}$  inducing an equivalence  $\Phi : \mathbb{P}^{n.c.}(K_{\delta^{-1}} \otimes_K V \otimes_K K_{\epsilon}) \longrightarrow \mathbb{P}^{n.c.}(W^{i*})$  and an integer  $i$  such that*

$$F \cong [-i] \circ \Phi \circ T_{\delta, \epsilon}.$$

*Furthermore,  $\delta, \epsilon$  and  $i$  are unique up to natural equivalence, while  $\Phi$  is naturally equivalent to  $\Phi'$  if and only if there exist nonzero  $a, b \in K$  such that  $\phi' \phi^{-1}(w) = a \cdot w \cdot b$  for all  $w \in W^{i*}$ .*

## 4.1. Applications to arithmetic noncommutative projective lines.

4.1.1.  $\mathbb{S}^{n.c.}(V)$  is a domain. In [9, Theorem 3.7], homological techniques are employed to prove that if  $V$  is a  $K - K$ -bimodule of left-right dimension  $(2, 2)$ , then  $\mathbb{S}^{n.c.}(V)$  is a domain in the sense that if  $x \in \mathbb{S}^{n.c.}(V)_{ij}$  and  $y \in \mathbb{S}^{n.c.}(V)_{jl}$  then  $xy = 0$  implies that  $x = 0$  or  $y = 0$ . Theorem 1.1 allows us to produce a much shorter proof of a generalization.

**Proposition 4.2.** *Suppose  $K$  and  $L$  are finite extensions of  $k$  and  $N$  is a  $k$ -central  $K - L$ -bimodule of left-right dimension  $(2, 2)$ . Then  $\mathbb{S}^{n.c.}(N)$  is a domain in the sense above.*

*Proof.* Theorem 1.1 implies it is sufficient to prove the result for the isomorphic ring  $H$ . But in this case the result follows immediately from [7, Lemma 1.3].  $\square$



**4.1.2. The Bondal-Orlov Theorem.** It is known that homogeneous noncommutative curves of genus zero have a Bondal-Orlov type reconstruction theorem. Therefore, it follows from Corollary 1.2 that the same holds for arithmetic noncommutative projective lines. Since we could not find a reference, we include a proof of this fact below. To this end, we recall from [6, Section 3] that the repetitive category of an abelian category  $\mathbf{A}$  is the additive closure of the union of disjoint copies of  $\mathbf{A}$  (labeled  $\mathbf{A}[n]$  for  $n \in \mathbb{Z}$  with objects labeled  $A[n]$ ) with morphisms given by  $\text{Hom}(A[m], B[n]) = \text{Ext}_{\mathbf{A}}^{n-m}(A, B)$  and composition given by Yoneda product of extensions. The notation for the second part of the next theorem is defined before, and in the statement of, Theorem 4.1.

**Theorem 4.3.** *Let  $\mathbf{H}_1$  and  $\mathbf{H}_2$  be homogeneous noncommutative curves of genus zero. If there is a  $k$ -linear triangulated equivalence  $F : D^b(\mathbf{H}_1) \rightarrow D^b(\mathbf{H}_2)$ , then  $F \cong T^i \circ G$  where  $G$  is induced by a  $k$ -linear equivalence  $\underline{G} : \mathbf{H}_1 \rightarrow \mathbf{H}_2$  and  $T$  is the translation functor in the derived category. In particular, if  $\tilde{\mathbf{H}}_1 = \mathbb{P}^{n.c.}(V)$ ,  $\tilde{\mathbf{H}}_2 = \mathbb{P}^{n.c.}(W)$ ,  $k$  is perfect and  $\text{char } k \neq 2$ , then  $\underline{G} \cong [-i] \circ \Phi \circ T_{\delta, \epsilon}$ .*

*Proof.* By [6, Theorem 3.1], we may identify  $D^b(\mathbf{H}_1)$  and  $D^b(\mathbf{H}_2)$  with their repetitive category. Furthermore, since  $\mathbf{H}_1$  and  $\mathbf{H}_2$  are noetherian hereditary categories having Serre duality, we know that there exists an integer  $i$  such that all objects from  $\mathbf{H}_1[0]$  may be identified, via  $F$ , as objects in the additive closure of  $\mathbf{H}_2[i]$  and  $\mathbf{H}_2[i+1]$  in the repetitive category of  $\mathbf{H}_2$  [6, Proposition 7.2]. Therefore, composing  $F$  with an iterate of the translation functor, we may assume objects from  $\mathbf{H}_1[0]$  may be identified with objects in the additive closure of  $\mathbf{H}_2[0]$  and  $\mathbf{H}_2[1]$ . By definition of morphism in the repetitive category,

$$(4-1) \quad \text{Hom}(\mathbf{H}_2[1], \mathbf{H}_2[0]) = 0.$$

We claim that either the image of all objects of  $\mathbf{H}_1[0]$  are in  $\mathbf{H}_2[0]$  or the image of all objects of  $\mathbf{H}_1[0]$  are in  $\mathbf{H}_2[1]$ . Given the claim, then by composing with a translation we may assume the former is true, so that there is an integer  $j$  such that  $T^j \circ F$  restricts to a  $k$ -linear equivalence from  $\mathbf{H}_1$  to  $\mathbf{H}_2$ . The result will follow. Therefore, to complete the proof, we prove the claim.

Since  $F$  preserves indecomposability, every indecomposable of  $\mathbf{H}_1[0]$  is sent to an indecomposable in  $\mathbf{H}_2[0]$  or an indecomposable in  $\mathbf{H}_2[1]$ . We will use this fact without comment throughout the proof. We will also use the notation introduced by (3-11).

*Step 1: We show that if  $\mathcal{S} \in \mathbf{H}_1[0]$  is simple, then  $F(\mathcal{S})$  has finite length.* For, if not, then  $F(\mathcal{S}) \cong \mathcal{O}(j)$  for some  $j \in \mathbb{Z}$  by [7, Proposition 1.1]. On the other hand, since  $F$  commutes with the Serre functor and translations, it commutes with all powers of  $\tau$ , so we have

$$\begin{aligned} \mathcal{O}(j) &\cong F(\mathcal{S}) \\ &\cong F(\tau\mathcal{S}) \\ &\cong \tau F(\mathcal{S}) \\ &\cong \mathcal{O}(j-2), \end{aligned}$$

which contradicts [9, Corollary 3.12].

*Step 2: We show that if  $F(\mathcal{O}(j)) \in \mathbf{H}_2[1]$  for some  $j$ , then the image of  $\mathbf{H}_1[0]$  is in  $\mathbf{H}_2[1]$ .* Since  $\text{Hom}_{\mathbf{H}_1}(\mathcal{O}(j), \mathcal{S}) \neq 0$  for all simples  $\mathcal{S} \in \mathbf{H}_1$  by [7, Proposition 1.10(b)], it follows from (4-1) that  $F(\mathcal{S}) \in \mathbf{H}_2[1]$  for all simples  $\mathcal{S}$ , and thus that

$F(\mathcal{T}) \in \mathbf{H}_2[1]$  for all finite length  $\mathcal{T}$  in  $\mathbf{H}_1$ . If  $F(\mathcal{O}(l)) \in \mathbf{H}_2[0]$  for some  $l$ , then

$$\begin{aligned} \mathrm{Hom}_{D^b(\mathbf{H}_2)}(F(\mathcal{O}(l)), F(\mathcal{S})) &\cong \mathrm{Ext}_{\mathbf{H}_2}^1(F(\mathcal{O}(l)), F(\mathcal{S})) \\ &\cong D\mathrm{Hom}_{\mathbf{H}_2}(F(\mathcal{S}), F(\mathcal{O}(l-2))) \\ &= 0 \end{aligned}$$

where the second isomorphism follows from Serre duality and from the fact that  $F$  commutes with translations and the Serre functor. This contradicts the fact that  $\mathrm{Hom}_{\mathbf{H}_1}(\mathcal{O}(l), \mathcal{S}) \neq 0$ , and Step 2 follows.

*Step 3:* We show that if  $F(\mathcal{O}(j)) \in \mathbf{H}_2[0]$  for all  $j$  then the image of  $\mathbf{H}_1[0]$  is in  $\mathbf{H}_2[0]$ . First, suppose  $F(\mathcal{O}(l))$  is a line bundle for some  $l$ . If  $F(\mathcal{S}) \in \mathbf{H}_2[1]$  for some simple  $\mathcal{S} \in \mathbf{H}_1$ , then  $0 \neq \mathrm{Hom}_{\mathbf{H}_1}(\mathcal{O}(l), \mathcal{S})$ . On the other hand,

$$\mathrm{Hom}_{D^b(\mathbf{H}_2)}(F(\mathcal{O}(l)), F(\mathcal{S})) \cong \mathrm{Ext}_{\mathbf{H}_2}^1(F(\mathcal{O}(l)), F(\mathcal{S})) = 0$$

where the last equality follows from Serre duality and Step 1. Step 3 follows in this case. Next, suppose  $F(\mathcal{O}(j))$  has finite length for all  $j$ . Then, since  $F$  commutes with translation, Step 1 implies that the image of  $F$  contains no line bundles, which contradicts the fact that  $F$  is an equivalence.

The second part of the theorem follows from the first and Theorem 4.1.  $\square$

#### 4.2. Applications to homogeneous noncommutative curves of genus zero.

We now describe our contributions to Questions 1.3, 1.4, and 1.5. For the remainder of this section, we assume  $\mathbf{H}$  is a homogeneous noncommutative curve of genus zero with underlying  $(2, 2)$ -bimodule  $M$  and with  $\mathrm{End}(\mathcal{L}) = \mathrm{End}(\overline{\mathcal{L}})$  commutative. We also assume  $k$  is perfect and  $\mathrm{char} k \neq 2$  so that we may invoke Theorem 4.1. Thus, by Corollary 1.2, we may identify  $\mathbf{H}$  with  $\mathbb{P}^{n.c.}(M)$ .

We will also use the following notation: if  $\mathrm{Gr}^{\mathbb{S}^{n.c.}}(M)$  is the category of graded right  $\mathbb{S}^{n.c.}(M)$ -modules, then we let  $\pi : \mathrm{Gr}^{\mathbb{S}^{n.c.}}(M) \rightarrow \mathbb{P}^{n.c.}(M)$  denote the quotient functor.

4.2.1. *Question 1.3.* Our contribution towards Question 1.3 is the following:

**Proposition 4.4.** *The space  $\mathbf{H}$  has a  $k$ -linear autoequivalence sending  $\mathcal{L}$  to  $\overline{\mathcal{L}}$  if and only if  $M \cong K_\sigma \otimes {}^*M \otimes K_\epsilon$  for some  $\sigma, \epsilon \in \mathrm{Gal}(K/k)$ . This is the case if and only if the group of  $k$ -linear autoequivalences acts transitively on the set of line bundles in  $\mathbf{H}$*

*Proof.* By [9, Lemma 3.16 and Corollary 3.17], we must have  $\mathcal{L} = \pi e_j \mathbb{S}^{n.c.}(M)$  for some integer  $j$ . Since  $M$  is a  $(2, 2)$ -bimodule, it follows that  $\overline{\mathcal{L}} = \pi e_{j-1} \mathbb{S}^{n.c.}(M)$ .

Now suppose  $F$  is a  $k$ -linear equivalence satisfying the hypothesis. By Theorem 4.1,  $F$  is a composition

$$\mathbb{P}^{n.c.}(M) \xrightarrow{T_{\delta, \epsilon}} \mathbb{P}^{n.c.}(K_{\delta^{-1}} \otimes M \otimes K_\epsilon) \xrightarrow{\Phi} \mathbb{P}^{n.c.}(M^{i*}) \xrightarrow{[-i]} \mathbb{P}^{n.c.}(M).$$

Since  $F(\mathcal{L}) \cong \overline{\mathcal{L}}$ , it follows from [9, Lemma 4.1, Lemma 4.2(1) and Lemma 4.8(1)] that  $i = -1$ . Therefore,  $\Phi$  is induced by an isomorphism  $K_{\delta^{-1}} \otimes M \otimes K_\epsilon \rightarrow {}^*M$  as desired.

Conversely, if there is an isomorphism  $\phi : K_{\delta^{-1}} \otimes M \otimes K_\epsilon \rightarrow {}^*M$ , then the equivalence

$$\mathbb{P}^{n.c.}(M) \xrightarrow{T_{\delta, \epsilon}} \mathbb{P}^{n.c.}(K_{\delta^{-1}} \otimes M \otimes K_\epsilon) \xrightarrow{\Phi} \mathbb{P}^{n.c.}({}^*M) \xrightarrow{[1]} \mathbb{P}^{n.c.}(M)$$

such that  $\Phi$  is induced by  $\phi$  sends  $\mathcal{L}$  to  $\overline{\mathcal{L}}$  by [9, Lemma 4.1, Lemma 4.2(1) and Lemma 4.8(1)].

The last result follows from the fact that every line bundle in  $\mathbf{H}$  is either of the form  $\tau^i \mathcal{L}$  for some  $i$  or of the form  $\tau^i \overline{\mathcal{L}}$  for some  $i$ .  $\square$

4.2.2. *Question 1.4.* In order to address Question 1.4, we need the following result.

**Lemma 4.5.** *The automorphism group  $\text{Aut}(\mathbb{X})$  of  $\mathbf{H}$  (defined in Section 1.3) is isomorphic to the group of shift-free  $k$ -linear autoequivalences of  $\mathbb{P}^{n.c.}(M)$ .*

*Proof.* If  $F$  is a  $k$ -linear autoequivalence of  $\mathbb{P}^{n.c.}(M)$ , then by Theorem 4.1 (and in the notation introduced before, and in, the statement of the theorem),  $F \cong [-i] \circ \Phi \circ T_{\delta, \epsilon}$ . Since  $\mathcal{L}$  is of the form  $\pi e_j \mathbb{S}^{n.c.}(M)$ , it follows from [9, Lemma 4.1, Lemma 4.2(1) and Lemma 4.8(1)] that  $F(\mathcal{L}) \cong \pi e_{j+i} \mathbb{S}^{n.c.}(M)$ . Thus,  $F(\mathcal{L}) \cong \mathcal{L}$  if and only if  $i = 0$ , i.e. if and only if  $F$  is shift-free.  $\square$

The group of shift-free autoequivalences of  $\mathbb{P}^{n.c.}(M)$ ,  $\text{Aut } \mathbb{P}^{n.c.}(M)$ , is described in [9]. For the readers convenience, we recount the result here. In order to proceed, we describe some notation: let  $\text{Stab } M$  denote the subgroup of  $\text{Gal}(K/k)^2$  consisting of  $(\delta, \epsilon)$  such that  $K_{\delta^{-1}} \otimes_K M \otimes_K K_{\epsilon} \cong M$  and let  $\text{Aut } M$  denote the group (under composition) of bimodule isomorphisms  $M \rightarrow M$  modulo the relation defined by setting  $\phi' \equiv \phi$  if and only if there exist nonzero  $a, b \in K$  such that  $\phi' \phi^{-1}(m) = a \cdot m \cdot b$  for all  $m \in M$ .

We will also need to recall some preliminaries regarding simple bimodules. In [10], simple left finite dimensional  $k$ -central  $K - K$ -bimodules are classified. Since this classification will be invoked in what follows, we recall it now. Let  $\overline{K}$  be an algebraic closure of  $K$ , let  $\text{Emb}(K)$  denote the set of  $k$ -algebra maps  $K \rightarrow \overline{K}$ , and let  $G = \text{Gal}(\overline{K}/K)$ . Now,  $G$  acts on  $\text{Emb}(K)$  by left composition. Given  $\lambda \in \text{Emb}(K)$ , we denote the orbit of  $\lambda$  under this action by  $\lambda^G$ . We denote the set of finite orbits of  $\text{Emb}(K)$  under the action of  $G$  by  $\Lambda(K)$ .

**Theorem 4.6.** [10] *There is a one-to-one correspondence between  $\Lambda(K)$  and isomorphism classes of simple left finite dimensional  $k$ -central  $K - K$ -bimodules.*

In fact, the class of the simple bimodule corresponding to  $\lambda$  is the class of the bimodule,  $V(\lambda)$ , defined as follows: as a set,  $V(\lambda) = K \vee \lambda(K)$ , and this set with its usual field structure is denoted  $K(\lambda)$ . The left and right action are via multiplication in this field:  $x \cdot v := xv$  while  $v \cdot x := v\lambda(x)$  ([3, Proposition 2.3]).

The group  $\text{Aut } \mathbb{P}^{n.c.}(M)$  is described in the following

**Theorem 4.7.** *There is a group homomorphism*

$$\theta : \text{Stab}(M)^{op} \rightarrow \text{Aut}(\text{Aut}(M))$$

(described below) such that

$$\text{Aut } \mathbb{P}^{n.c.}(M) \cong \text{Aut } M \rtimes_{\theta} \text{Stab } M^{op}.$$

Furthermore, the factors of  $\text{Aut } \mathbb{P}^{n.c.}(M)$  are described as follows:

- (1) If  $M \cong K_{\sigma} \oplus K_{\epsilon}$ , where  $\sigma, \epsilon \in \text{Gal}(K/k)$  and  $\sigma \neq \epsilon$ , then

$$\text{Aut } M \cong K^* \times K^* / \{(a\sigma(b), a\epsilon(b)) \mid a, b \in K^*\}.$$

and

$$\text{Stab } M = \{(\delta, \gamma) \mid \{\delta^{-1}\sigma\gamma, \delta^{-1}\epsilon\gamma\} = \{\sigma, \epsilon\}\},$$

- (2) If  $M \cong V(\lambda)$ , and if, for  $\delta \in \text{Gal}(K/k)$ ,  $\bar{\delta}$  denotes an extension of  $\delta$  to  $\bar{K}$ , then

$$\text{Aut}(M) \cong K(\lambda)^*/K^*\lambda(K)^*.$$

and

$$\text{Stab}(M) = \{(\delta, \gamma) \in \text{Gal}(K/k)^2 \mid (\bar{\delta}^{-1}\lambda\gamma)^G = \lambda^G\}.$$

To describe  $\theta$ , we will need to recall [9, Lemma 2.2]:

**Lemma 4.8.** *Suppose  $\sigma, \epsilon \in \text{Gal}(K/k)$  and let  $\bar{\sigma}$  denote an extension of  $\sigma$  to  $\bar{K}$ . Under the isomorphism given in Theorem 4.6, the  $G$ -orbit of  $\bar{\sigma}\lambda\epsilon$  corresponds to the isomorphism class of the simple bimodule  $K_\sigma \otimes V(\lambda) \otimes K_\epsilon$ .*

According to [9, Lemma 2.1], there are only two possibilities for  $M$  such that  $\mathbb{P}^{n.c.}(M)$  is not equivalent to the commutative projective line. We define  $\theta$  according to the corresponding structure of  $M$ .

- (1) In case  $\sigma, \epsilon \in \text{Gal}(K/k)$ ,  $\sigma \neq \epsilon$ ,  $M = K_\sigma \oplus K_\epsilon$ , and  $(\delta, \gamma) \in \text{Stab } M$  is such that  $\delta^{-1}\sigma\gamma = \sigma$ , we define  $\theta(\delta, \gamma)$  as the element in

$$\text{Aut}(K^* \times K^* / \{(a\sigma(b), a\epsilon(b)) \mid a, b \in K^*\})$$

which sends the class of the pair  $(c, d) \in K^* \times K^*$  to the class of the pair  $(\delta^{-1}(c), \delta^{-1}(d))$ . In other words,  $\theta(\delta, \gamma)$  acting on the class of an isomorphism  $\phi$  equals the class of  $\delta^{-1}\phi\delta$ , where  $\delta^{-1}$  and  $\delta$  act coordinate-wise on  $K^2$ .

If  $\delta^{-1}\sigma\gamma = \epsilon$ , then we define  $\theta(\delta, \gamma)$  as automorphism sending the class of  $(c, d)$  to the class of  $(\delta^{-1}(d), \delta^{-1}(c))$ . In other words,  $\theta(\delta, \gamma)$  acting on the class of an isomorphism  $\phi$  equals the class of  $v\delta^{-1}\phi\delta v$ , where  $\delta^{-1}$  and  $\delta$  act coordinate-wise on  $K^2$  and  $v$  is the linear transformation that exchanges the coordinates.

- (2) In case  $M = V(\lambda)$  and  $(\delta, \gamma) \in \text{Stab } M$ , we define  $\theta(\delta, \gamma)$  as the function sending the class of  $c \in K(\lambda)^*$  to the class of  $\psi^{-1}(c)$ , where  $\psi : K(\lambda) \rightarrow K(\lambda)$  is the  $k$ -algebra isomorphism defined in [9, Proposition 7.1]. Specifically, there exists some extension of  $\delta$ ,  $\bar{\delta}$ , such that  $\bar{\delta}^{-1}\lambda\gamma = \lambda$  by Lemma 4.8. We let  $\psi = \bar{\delta}|_{K(\lambda)}$ .

Thus,  $\theta(\delta, \gamma)$  acting on the class of an isomorphism  $\phi$  equals the class of  $\psi^{-1}\phi\psi$ .

**4.2.3. Question 1.5.** We now describe the Auslander-Reiten translation functor of  $H$ . As described in [9], it is simply the equivalence

$$\mathbb{P}^{n.c.}(M) \xrightarrow{\Phi} \mathbb{P}^{n.c.}(M^{**}) \xrightarrow{[-2]} \mathbb{P}^{n.c.}(M)$$

where  $\Phi$  is induced by an isomorphism  $\phi : M \rightarrow M^{**}$  (which exists, for example, by [3, Theorem 3.13]). This address Question 1.5.

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